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A FULL-INFORMATION BEST-CHOICE PROBLEM WITH ALLOWANCE

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0. INTRODUCTION

The basic form of the full-information best-choice problem, originally studied by Gilbert and Mosteller(1966) can be described as follows: Let X_i, $i=1, 2, \ldots, n$ be the value attached to the i^{th} item and suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables from a known continuous distribution function F. On arrival of the i^{th} item, we observe X_i and decide immediately either to accept or reject this item, weighing the possibility of obtaining a better item against the risk of losing the current item. The objective is to maximize the probability of choosing the overall best, i.e., the item which has the largest value among all, assuming no solicitation of the previously rejected item. If the n-1 items have been rejected, the last one must be accepted. Generalizations and extensions of this problem were made by Petruccelli(1982) and Tamaki(1986).

In the real situation, though the chosen item is not the overall best, we will be satisfied with it if its value is sufficiently large compared with the overall best. This motivates our problems. In Section 1, an allowance function $\rho(.)$ will be introduced. Let x be the value of the chosen item and y be the largest value among all, then this selection is a *success* if $x \geq y - \rho(y)$. We seek an optimal strategy, which maximizes the probability of success.

1. ALLOWANCE MODEL

Here $X_1, X_2, \ldots, X_n$ are assumed to be independent and identically distributed non-negative random variables from a known continuous distribution function $F(x)$. $F(x)$ is also assumed to be increasing on the set where $0 < F(x) < 1$.

Let $Y_j = \max\{X_1, X_2, \ldots, X_j\}$, $1 \leq j \leq n$, and $\rho(y)$ be a prescribed allowance function defined on $[0,\infty)$. Then the state of the decision process after having observed $X_1, X_2, \ldots, X_{n-k}$ can be described as (x, y, k), $0 \leqq x \leqq y$, $0 \leqq k < n$, if $X_{n-k}=x$ and $Y_{n-k}=y$ (note that k represents the number of the remaining observations available) and choosing X_{n-k} in this state can be regarded as a success if $Y_n - \rho(Y_n) \leqq X_{n-k} \leqq Y_n$.

To make the subsequent analysis simple, we put the following two assumptions on $\rho(y)$.

(A1) $\rho(y)$ is a continuous function of y with $0 \leqq \rho(y) \leqq y$.

(A2) $y-\rho(y)$ is non-decreasing in y.

(A1) is a natural assumption. Now let

$$E=\{(x, y) : y-\rho(y) \leqq x \leqq y\},$$

then (A2) assures that, for each k, it is not optimal to accept X_{n-k} in state (x, y, k) for which $(x, y) \notin E$. This is easily seen because, under (A2), if $(x, y) \notin E$ then $(x, y') \notin E$ for $y' \geqq y$ and because the maximum value observsd so far does not decrease as time goes. This is why we confine our attention to state (x, y, k) for which $(x, y) \in E$. If $(x, y) \in E$, x is called a *candidate* with respect to y (sometimes x is simply called a candidate). It should be noted that

under (A2), if x is a candidate with respect to y, then x is also a candidate with respect to y' when $(x \leqq) y' < y$. (1.1)

Define

$$\beta(x)=\sup\{y : (x, y) \in E\}.$$

$\beta(x)$ then represents the maximal value of y, for which x remains a candidate. It follows from (A1) and (A2) that $\beta(x)$ is increasing in x where $F(\beta(x))<1$, but possibly have several discontinuity points. To guarantee that $\beta(x)$ is a continuously increasing function, (A2) must be replaced by

(A2)' $y-\rho(y)$ is increasing in y.

Two typical cases of $\rho(y)$ which satisfy (A1) and (A2) are as follows. Corresponding $\beta(x)$ is also given:

Example 1 (proportional allowance)

$$\rho(y)=ry, \quad y\geqq 0$$
$$\beta(x)=x/\bar{r}, \quad x\geqq 0 \quad \text{where } \bar{r}=1-r \text{ and } 0\leqq r<1.$$

Example 2 (constant allowance)

$$\rho(y)=\min(y,c), \quad y\geqq 0 \text{ and } c>0$$
$$\beta(x)=x+c, \quad x\geqq 0.$$

Example 1 satisfies (A2)' but Example 2 does not satisfy (A2)'.

In state (x,y,k), we have two alternatives; acceptance (stopping) and rejection (continuance). Let $s_k(x,y)$ be the probability of success when we accept the candidate and $c_k(x,y)$ be the corresponding probability when we assume continuation in an optimal manner. It is easy to see that $s_k(x,y)$ $(c_k(x,y))$ depends on (x,y) only through x (y). So we simply write $s_k(x)$ and $c_k(y)$ for them. Put, for $(x,y)\in E$ and $0\leqq k<n$,

$$v_k(x,y)=\max\{s_k(x),\ c_k(y)\}. \tag{1.2}$$

Then we have the following recursive relations

$$s_k(x)=\{F(\beta(x))\}^k, \qquad 0\leqq k<n \tag{1.3}$$

$$c_k(y)=F(y-\rho(y))\,c_{k-1}(y)+\int_{y-\rho(y)}^{y} v_{k-1}(t,y)\,dF(t)+\int_{y}^{\infty} v_{k-1}(t,t)\,dF(t) \tag{1.4}$$
$$1\leqq k<n,$$

with the boundary condition $c_0(y)\equiv 0$.

Eq. (1.3) is immediate since all the remaining observations must have values not greater than $\beta(x)$ for our selection X_{n-k} to be a success. After leaving state (x,y,k), X_{n-k+1} is observed but rejected if it is not a candidate, i.e.,

$X_{n-k+1}<y-\rho(y)$. Otherwise state makes transition into $(t, y, k-1)$ or $(t, t, k-1)$ depending on whether $X_{n-k+1}=t<y$ or $X_{n-k+1}=t\geqq y$. This yields Eq. (1.4). Note that Eq. (1.2) for $k=n-1$ is defined only for $x=y$ due to $X_1=Y_1$ and the probability of success is calculated as $\int_0^\infty v_{n-1}(x, x)\,dF(x)$.

We start with the following lemma.

Lemma 1.1. Assume that (A1) and (A2) hold. Then, for each k,

$c_k(y)$ is continuous and non-increasing in y.

Proof. Denote by σ_i, $i=1, 2$, an optimal strategy followed after leaving state (x, y_i, k), where $y_1>y_2(\geqq x)$ and compare the following two situations:

situation 1: We leave (x, y_1, k) and use σ_1.

situation 2: We leave (x, y_2, k) and use σ_1.

It is easy to see from (1.1) that the success in situation 1 is also a success in situation 2. Thus the probability of success in situation 2 is at least as large as $c_k(y_1)$, and consequently $c_k(y_2)\geqq c_k(y_1)$. Continuity follows by induction on k from (1.4).

Remark. We can prove $dc_k(y)/dy\leqq 0$ by induction on k, assuming all differentiability required. Let $f(t)=dF(t)/dt$, then differentiating formally both sides of (1.4) yields

$$
\begin{aligned}
c_k'(y) &= F(y-\rho(y))\,c_{k-1}'(y) \\
&\quad + (\rho'(y)-1)\,f(y-\rho(y))\,\{v_{k-1}(y-\rho(y), y)-c_{k-1}(y)\} \\
&\quad + \int_{y-\rho(y)}^{y} \{\partial v_{k-1}(t, y)/\partial y\}\,dF(t).
\end{aligned}
$$

$c_{k-1}(y)$ and $v_{k-1}(t, y)$ are non-increasing in y from the induction hypothesis and $1-\rho'(y)\geqq 0$ from (A2). Hence, each term in the right side of the above equation is non-positive and $c_k'(y)\leqq 0$.

Let, for $0 \leqq k < n$,

$$G_k = \{(x, y) \in E:\ s_k(x) \geqq c_k(y)\}.$$

Then it is optimal to accept the candidate in state (x, y, k) for which $(x, y) \in G_k$. Since, under (A1) and (A2), $s_k(x)$ is increasing in x, where $s_k(x) < 1$ (because $\beta(x)$ is increasing), the following theorem is an immediate consequence from Lemma 1.1.

Theorem 1.2. Assume that (A1) and (A2) hold. Then, for $k \geqq 1$, there exist two critical numbers

$$a_k = \inf\{y:\ s_k(y) \geqq c_k(y)\}, \tag{1.5}$$

$$b_k = \inf\{y:\ s_k(y - \rho(y)) \geqq c_k(y)\}, \tag{1.6}$$

and a non-increasing continuous function

$$\wp_k(y) = \inf\{x:\ s_k(x) \geqq c_k(y)\},\quad a_k \leqq y \leqq b_k, \tag{1.7}$$

such that

$$G_k = \{(x, y):\ \wp_k(y) \leqq x \leqq y,\ a_k \leqq y \leqq b_k\} \cup \{(x, y):\ y - \rho(y) \leqq x \leqq y,\ b_k < y\}.$$

When $k=0$, $a_0 = b_0 = 0$ and $G_0 = E$.

Remarks. (1) If X_i is bounded, i.e., there exists A such that $F(t) = 1$ for $t \geqq A$, then $a_k \leqq A - \rho(A)$ since $s_k(x) \equiv 1$ for $x \geqq A - \rho(A)$.

(2) Assume that (A1) and (A2)' hold. Then $\beta(x)$ is continuously increasing and so is $s_k(x)$, where $s_k(x) < 1$. Hence, in this case, (1.5)-(1.7) can be reduced to the following forms:

a_k is the unique root y of the equation

$$s_k(y) = c_k(y)\ , \tag{1.5$'$}$$

b_k is the unique root y of the equation

$$s_k(y - \rho(y)) = c_k(y)\ , \tag{1.6$'$}$$

and $\mathscr{S}_k(y)$ is the unique root x of the equation

$$s_k(x)=c_k(y),\quad a_k\leqq y\leqq b_k\ . \tag{1.7)'}$$

Lemma 1.3.. Assume that (A1) and (A2)' hold. Then

$$G_{k+1}\subseteq G_k,\quad 0\leqq k<n-1.$$

Proof. Let $k^*>0$ be the smallest k such that $G_k\neq E$. Then, by the continuity property of $s_k(x)$ and $c_k(y)$, there exists a non-empty set (subset of G_k) defined by

$$\overline{G}_k=\{(x,y)\in G_k:\ x=\mathscr{S}_k(y)\ \},\quad k\geqq k^*\ .$$

To prove the lemma, it is sufficient to show that, for $(x,y)\in\overline{G}_k$,

$$c_{k+1}(y)\geqq s_{k+1}(x). \tag{1.8}$$

From (1.4) and the assumption that $(x,y)\in\overline{G}_k$,

$$\begin{aligned}c_{k+1}(y)&=F(y-\rho(y))c_k(y)+\int_{y-\rho(y)}^{x}v_k(t,y)dF(t)\\&\quad+\int_x^y v_k(t,y)dF(t)+\int_y^\infty v_k(t,t)dF(t)\\&=F(y-\rho(y))c_k(y)+\int_{y-\rho(y)}^{x}c_k(y)dF(t)+\int_x^\infty s_k(t)dF(t)\\&=F(x)c_k(y)+\int_x^\infty s_k(t)dF(t)\\&=F(x)s_k(x)+\int_x^\infty s_k(t)dF(t),\end{aligned}$$

where the last equality follows since $c_k(y)=s_k(x)$ on $(x,y)\in\overline{G}_k$. Thus, from the monotonicity property of $s_k(t)$ with respect to t,

$$\begin{aligned}&c_{k+1}(y)-s_{k+1}(x)\\&=F(x)s_k(x)+\int_x^\infty s_k(t)dF(t)-s_{k+1}(x)\\&\geqq F(x)s_k(x)+s_k(x)\int_x^\infty dF(x)-s_{k+1}(x)\\&=s_k(x)-s_{k+1}(x)\\&=\{1-F(\beta(x))\}\{F(\beta(x))\}^k\\&\geqq 0,\end{aligned}$$

which proves (1.8).

Remark. Example 2 (constant allowance) does not satisfy (A2)$'$, but it is easy to show by induction that a_k, b_k, and $\mathscr{S}_k(y)$ can be determined by (1.5)$'$ – (1.7)$'$ and Lemma 1.3 still holds if $F(2c)<1$. We can achieve success with certainty if $F(2c)=1$. In this case, there exists a finite number A such that $A=\inf\{x:\ F(t)=1,\ t\geqq x\}\leqq 2c$. Hence, we employ a strategy which accepts an item whose value exceeds $A-c$ and, if no such item appears in the first $n-1$ observations, accepts the last item.

What is left is to determine the sequences of the decision numbers $\{a_k\}$ and $\{b_k\}$, and the sequence of the decision function $\mathscr{S}_k(y)$ for $a_k\leqq y\leqq b_k$. Hereafter we assume (A1) and (A2)$'$, unless otherwise specified. Letting $k=1$ in (1.3) and (1.4) yields

$$s_1(x)=F(\beta(x)),$$

$$c_1(y)=1-F(y-\rho(y)).$$

Thus, from (1.5)$'$ – (1.7)$'$, a_1 is the unique root y of the equation

$$F(\beta(y))+F(y-\rho(y))=1, \tag{1.9}$$

b_1 is the unique root y of the equation

$$F(y)+F(y-\rho(y))=1, \tag{1.10}$$

and

$$\mathscr{S}_1(y)=F^{-1}(c_1(y))-\rho(F^{-1}(c_1(y))). \tag{1.11}$$

For $k\geqq 2$, corresponding quantities are difficult to be obtained by solving recursively (1.2) – (1.4). Repeated use of (1.4) yields, for $y>0$,

$$c_k(y)=\Sigma_{j=1}^{k}[F(y-\rho(y))]^{j-1}\{\textstyle\int_{y-\rho(y)}^{y} v_{k-j}(t,y)\,dF(t)+\int_{y}^{\infty} v_{k-j}(t,t)\,dF(t)\}.$$

In particular, for $y\geqq b_{k-1}$,

$$c_k(y)=\Sigma_{j=1}^{k}[F(y-\rho(y))]^{j-1}\textstyle\int_{y-\rho(y)}^{\infty}\{F(\beta(t))\}^{k-j}\,dF(t), \tag{1.12}$$

which follows because the optimal strategy, after leaving state (x, y, k), immediately accepts a candidate that appears due to the monotonicity property of

G_k given in Lemma 1.3. This makes it easy to calculate b_k.

Lemma 1.4. The decision number b_k, $1 \leqq k < n$, is the unique root $y (\geqq b_{k-1})$ of the equation

$$\{F(y)\}^k = \sum_{j=1}^{k} [F(y-\rho(y))]^{j-1} \int_{y-\rho(y)}^{\infty} \{F(\beta(t))\}^{k-j} dF(t). \tag{1.13}$$

Proof. From (1.6)′, b_k is the value of y which equates $s_k(y-\rho(y))$ with $c_k(y)$. Thus the result is immediate from (1.12), since $b_k \geqq b_{k-1}$ from Lemma 1.3 and $s_k(y-\rho(y)) = \{F(\beta(y-\rho(y)))\}^k = \{F(y)\}^k$ from the definition of $\beta(.)$.

Remark. Let $\rho(y) \equiv 0$ and denote by t_k the corresponding decision number b_k. Then, from (1.13), t_k satisfies

$$\{F(y)\}^k = \sum_{j=1}^{k} \{F(y)\}^{j-1} \int_{y}^{\infty} \{F(t)\}^{k-j} dF(t) \tag{1.13$'$}$$

or equivalently

$$\{F(y)\}^k = \sum_{j=1}^{k} [\{F(y)\}^{j-1} - \{F(y)\}^k] / (k-j+1).$$

This is the well known result in the full-information best-choice problem (see Gilbert and Mosteller 1966 or Sakaguchi 1973)

The following lemma provides an algorithm for calculating a_k and $\mathcal{P}_k(y)$ for $a_k \leqq y \leqq b_k$, when a_s, b_s, and $\mathcal{P}_s(y)$ for $a_s \leqq y \leqq b_s$ and $1 \leqq s < k$ are given. We use notation $I_j = (b_{j-1}, b_j]$, $j \geqq 1$.

Lemma 1.5. a_1, b_1, and $\mathcal{P}_1(y)$ for $a_1 \leqq y \leqq b_1$ are calculated from (1.9)-(1.11). Assume that a_s, b_s, and $\mathcal{P}_s(y)$ for $a_s \leqq y \leqq b_s$ are known, where $1 \leqq s < k$ $(2 \leqq k < n)$. First solve b_k from (1.13) and let i $(1 \leqq i < k)$ be the integer such that $a_{k-1} \in I_i$. Then $c_k(y)$ for $a_{k-1} \leqq y \leqq b_k$, a_k, and $\mathcal{P}_k(y)$ for $a_k \leqq y \leqq b_k$ can be calculated as follows:

(i) Let

$$\lambda_\ell(y)=\begin{cases}\prod_{s=\ell}^{k-1} F(\varphi_s(y)), & \ell<k\\ 1, & \ell=k\end{cases}$$

and define, for $i\leqq j\leqq k$,

$$I_j'=\begin{cases}(a_{k-1},\ b_i], & j=i\\ I_j, & i<j\leqq k.\end{cases}$$

Then, for $y\in I_j'$,

$$c_k(y)=\lambda_j(y)\sum_{\ell=0}^{j-1}\{F(y-\rho(y))\}^{j-1-\ell}\int_{y-\rho(y)}^{\infty}\{F(\beta(x))\}^{\ell}\,dF(x) \tag{1.14}$$

$$+\sum_{\ell=j}^{k-1}\lambda_{\ell+1}(y)\int_{\varphi_\ell(y)}^{\infty}\{F(\beta(x))\}^{\ell}\,dF(x),$$

where the vacuous sum is assumed to be 0.

(ii) a_k is the unique root y in $(a_{k-1}, b_k]$ of the equation

$$s_k(y)=c_k(y)\ .$$

(iii)

$$\varphi_k(y)=F^{-1}(\sqrt[k]{c_k(y)})-\rho(F^{-1}(\sqrt[k]{c_k(y)}))\ ,\quad a_k\leqq y\leqq b_k\ . \tag{1.15}$$

Proof. Fix $y\in I_j'$ for given j and define

$$d_\ell(y)=\begin{cases}\varphi_\ell(y), & j\leqq\ell\leqq k-1\\ y-\rho(y), & 0\leqq\ell\leqq j-1.\end{cases}$$

Then it is easily seen from Lemma 1.3 that, after leaving state (x, y, k), the optimal strategy immediately accepts $X_{n-\ell}$, $0\leqq\ell<k$, if $X_{n-\ell}\geqq d_\ell(y)$. Thus

$$c_k(y)=\sum_{\ell=0}^{k-1}\{\prod_{s=\ell+1}^{k-1}F(d_s(y))\}\int_{d_\ell(y)}^{\infty}s_\ell(x)\,dF(x),$$

which, combined with (1.3), yields (1.14). (ii) and (iii) are from (1.5)′ and (1.7)′.

Remark. It is easy to see from remark of Lemma 1.3 that Lemmas 1.4 and 1.5 hold for Example 2 (constant allowance) with $F(2c)<1$.

In principle, repeated use of Lemma 1.5 successively determines

a_1, b_1 and $\mathscr{S}_1(y)$ for $y\in[a_1, b_1]$

a_2, b_2 and $\mathscr{S}_2(y)$ for $y\in[a_2, b_2]$

$\vdots$

When F is a uniform distribution on [0, 1], some simplification can be done in calculating the decision numbers and the decision functions for Examples 1 and 2. Taking account of

$$\beta(t)=\begin{cases} t/\bar{r} & \text{(proportional allowance)} \\ t+c & \text{(constant allowance)} \end{cases},$$

we have, for $x<1-\rho(1)$,

$$\int_x^\infty \{F(\beta(t))\}^m dF(t)=\int_x^{1-\rho(1)}\{\beta(t)\}^m dt+\int_{1-\rho(1)}^1 dt$$

$$=\begin{cases} r+\bar{r}\{1-(x/\bar{r})^{m+1}\}/(m+1) & \text{(proportional allowance)} \\ c+\{1-(x+c)^{m+1}\}/(m+1) & \text{(constant allowance)} \end{cases}.$$

Hence, the following corollary is immediate from Lemmas 1.4 and 1.5.

Corollary 1.6. Assume that F(.) is a uniform distribution on [0, 1]. Let, for fixed k,

$$\lambda_\ell(y)=\begin{cases} \prod_{s=\ell}^{k-1} \mathscr{S}_s(y), & \ell<k \\ 1, & \ell=k \end{cases},$$

then (1.13)-(1.15) can be written as follows:

(i) Example 1 (proportional allowance)

b_k is the unique root y in $(b_{k-1}, 1]$ of the equation

$$y^k=\sum_{j=1}^k(\bar{r}y)^{k-j}\{r+\bar{r}(1-y^j)/j\}.$$

For $y \in I_j'$ $(i \leq j \leq k)$,

$$c_k(y) = \lambda_j(y) \sum_{\ell=0}^{j-1} (\bar{r}y)^{j-1-\ell} \{r + \bar{r}(1-y^{\ell+1})/(\ell+1)\}$$

$$+ \sum_{\ell=j}^{k-1} \lambda_{\ell+1}(y) \{r + \bar{r}[1-(\mathcal{G}_\ell(y)/\bar{r})^{\ell+1}]/(\ell+1)\}$$

and

$$\mathcal{G}_k(y) = \bar{r}[c_k(y)]^{1/k}.$$

(ii) Example 2 (constant allowance with $0<c<1/2$)

b_k is the unique root y in $(b_{k-1}, 1]$ of the equation

$$y^k = \sum_{j=1}^{k} (y-c)^{k-j} \{c + (1-y^j)/j\}.$$

For $y \in I_j'$ $(i \leq j \leq k)$,

$$c_k(y) = \lambda_j(y) \sum_{\ell=0}^{j-1} (y-c)^{j-1-\ell} \{c + (1-y^{\ell+1})/(\ell+1)\}$$

$$+ \sum_{\ell=j}^{k-1} \lambda_{\ell+1}(y) \{c + [1-(\mathcal{G}_\ell(y)+c)^{\ell+1}]/(\ell+1)\}$$

and

$$\mathcal{G}_k(y) = [c_k(y)]^{1/k} - c.$$

Note that the smallest possible value of x to be accepted in state (x, y, k) is $b_k - \rho(b_k)$. Hence, we may well conjecture that $t_k \geq b_k - \rho(b_k)$, $k \geq 1$, for any allowance function satisfying (A1) and (A2)$'$, where t_k as defined in (1.13)$'$ is the decision number of the corresponding non-allowance problem. However, this conjecture is not true. The following corollary gives an example for which $t_k < b_k - \rho(b_k)$ holds for some k.

Corollary 1.7. Let $\rho(y)$ be

$$\begin{cases} \rho(y) = 0, & x \leq d \\ \rho(y) > 0, & x > d \end{cases}$$

for fixed d such that $t_k \leq d$ for some $k \geq 2$. Then $t_k < b_k - \rho(b_k)$.

Proof. Define, for $x \geq 0$,

$$H_k(x)=\{F(\beta(x))\}^k-\textstyle\sum_{j=1}^{k}\{F(x)\}^{j-1}\int_x^\infty\{F(\beta(t))\}^{k-j}dF(t). \qquad (1.16)$$

Then, when $x\geqq b_{k-1}-\rho(b_{k-1})$, $H_k(x)$ can be expressed as

$$H_k(x)=s_k(x)-c_k(\beta(x)),$$

because substituting $y=\beta(x)$ into (1.12) yields

$$c_k(\beta(x))=\textstyle\sum_{j=1}^{k}\{F(x)\}^{j-1}\int_x^\infty\{F(\beta(t))\}^{k-j}dF(t),\quad x\geqq b_{k-1}-\rho(b_{k-1}).$$

Considering that, from Lemma 1.1, $H_k(x)$ is increasing in x when $x\geqq b_{k-1}-\rho(b_{k-1})$ and that $b_k-\rho(b_k)$ is, from (1.6)', the unique root x of the equation $s_k(x)=c_k(\beta(x))$, we find

$$H_k(x)\geqq 0,\quad x\geqq b_k-\rho(b_k).$$

Thus, to prove $t_k<b_k-\rho(b_k)$, it suffices to show $H_k(t_k)<0$. Since t_k satisfies, from (1.13)',

$$\{F(t_k)\}^{k-1}\int_{t_k}^\infty dF(t)=\{F(t_k)\}^k-\textstyle\sum_{j=1}^{k-1}\{F(t_k)\}^{j-1}\int_{t_k}^\infty\{F(t)\}^{k-j}dF(t), \qquad (1.17)$$

we have, from (1.16) and (1.17),

$$\begin{aligned}H_k(t_k)&=\{F(\beta(t_k))\}^k-\textstyle\sum_{j=1}^{k-1}\{F(t_k)\}^{j-1}\int_{t_k}^\infty\{F(\beta(t))\}^{k-j}dF(t)-\{F(t_k)\}^{k-1}\int_{t_k}^\infty dF(t)\\&=[\{F(\beta(t_k))\}^k-\{F(t_k)\}^k]-\textstyle\sum_{j=1}^{k-1}\{F(t_k)\}^{j-1}\int_{t_k}^\infty[\{F(\beta(t))\}^{k-j}-\{F(t)\}^{k-j}]dF(t)\\&=-\textstyle\sum_{j=1}^{k-1}\{F(t_k)\}^{j-1}\int_d^\infty[\{F(\beta(t))\}^{k-j}-\{F(t)\}^{k-j}]dF(t)\\&<0\quad(\text{when } k\geqq 2),\end{aligned}$$

where the last equality follows from $\beta(t)=t$ for $t\leqq d$ and the inequality follows from $\beta(t)>t$ for $t>d$.

It is of interest but difficult to investigate how the probability of success depends on the underlying distribution F and the allowance function employed. Before concluding this section, we make, for Examples 1 and 2, simple comparisons between a uniform distribution on [0, 1] and a triangular distribution on [0, 1] when n=2. Denote by P(Success|F) the probability of success under an optimal policy when the underlying distribution is F. We accept the first item if $X_1 \geqq a_1$, but continue and observe the second item if $X_1 < a_1$. Hence,

$$P(\text{Success}|F) = \int_0^{a_1} \{1 - F(t - \rho(t))\} dF(t) + \int_{a_1}^{\infty} F(\beta(t)) dF(t).$$

Let

$$F_U(x) = \begin{cases} x, & 0 \leqq x \leqq 1 \\ 1, & x > 1 \end{cases}$$

and

$$F_T(x) = \begin{cases} x^2, & 0 \leqq x \leqq 1 \\ 1, & x > 1. \end{cases}$$

We have, from straightforward calculation,

(i) Example 1 (proportional allowance)

$$P(\text{Success}|F_U) = 1 - \bar{r}/2 + \bar{r}/2(1+\bar{r}^2),$$

$$P(\text{Success}|F_T) = 1 - \bar{r}^2/2 + \bar{r}^2/2(1+\bar{r}^4),$$

$$\begin{aligned} P(\text{Success}|F_T) - P(\text{Success}|F_U) &= \bar{r}^3\{(1-\bar{r}^3) + (1-\bar{r})\bar{r}^4\}/2(1+\bar{r}^2)(1+\bar{r}^4) \\ &\geqq 0, \end{aligned}$$

and

(ii) Example 2 (constant allowance with $c < 1/2$)

$$P(\text{Success}|F_U) = 3/4 + c - c^2,$$

$$P(\text{Success}|F_T) = 3/4 + 4c/3 - 2c^2 + 4c^4/3,$$

$$\begin{aligned} P(\text{Success}|F_T) - P(\text{Success}|F_U) &= c(1+c)(1-2c)^2/3 \\ &\geqq 0. \end{aligned}$$

Does the inequality $P(\text{Success}|F_T) \geqq P(\text{Success}|F_U)$ correspond to the stochastic order relation $F_T(x) \leqq F_U(x)$? The answer is negative. Let

$$F_\lambda(x) = 1 - \exp(-\lambda x), \quad x \geqq 0 \text{ and } \lambda > 0.$$

Then, for the constant allowance case,

$$P(\text{Success}|F_\lambda) = 1 - \exp(-2\lambda c)/2\{\exp(\lambda c) + \exp(-\lambda c)\},$$

which is clearly increasing in λ. Thus $P(\text{Success}|F_{\lambda_1}) \geqq P(\text{Success}|F_{\lambda_2})$ goes together with $F_{\lambda_1}(x) \geqq F_{\lambda_2}(x)$ for $\lambda_1 > \lambda_2$.

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